

Local minimality of the volume-product at the simplex

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Abstract

It is proved that the simplex is a strict local minimum for the volume product, $\mathcal{P}(K) = \min_{z \in \text{int}(K)} |K| |K^z|$, in the Banach-Mazur space of n -dimensional (classes of) convex bodies. Linear local stability in the neighborhood of the simplex is proved as well. The proof consists of an extension to the non-symmetric setting of methods that were recently introduced by Nazarov, Petrov, Ryabogin and Zvavitch, as well as proving results of independent interest, concerning stability of square order of volumes of polars of non-symmetric convex bodies.

1 Introduction and Preliminaries

A *body* is a compact set which is the closure of its interior and, in particular, a *convex body* in \mathbb{R}^n is a compact convex set with nonempty interior. If K is a convex body in \mathbb{R}^n and z is an interior point of K , then the *polar body* K^z of K with center of polarity z is defined by

$$K^z = \{y \in \mathbb{R}^n : \langle y, x - z \rangle \leq 1 \text{ for all } x \in K\}$$

where $\langle \cdot, \cdot \rangle$ is the canonical scalar product in \mathbb{R}^n . In particular, if the center of polarity is taken to be the origin, we denote by K° the polar body of K and we clearly have $K^z = (K - z)^\circ$.

If A is a measurable set in \mathbb{R}^n and k is the minimal dimension of a flat containing A , we denote by $|A|$ the k -dimensional volume (Lebesgue measure) of A . There should

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be no confusion of the last notation with the notation for the Euclidean norm of a vector $x \in \mathbb{R}^n$ which is $|x| = \sqrt{\langle x, x \rangle}$. A well known result of Santaló [S] states that in every convex body K in \mathbb{R}^n there exists a unique point $s(K)$, called the *Santaló point* of K , such that

$$|K^{s(K)}| = \min_{z \in \text{int}(K)} |K^z|.$$

The *volume product* of K is defined by

$$\mathcal{P}(K) = \inf\{|K| |K^z| : z \in \text{int}(K)\}.$$

A well known conjecture, called sometimes Mahler's conjecture ([Ma1, Ma2]), states that, for every convex body K in \mathbb{R}^n ,

$$\mathcal{P}(K) \geq \mathcal{P}(S) = \frac{(n+1)^{n+1}}{(n!)^2} \quad (1)$$

where S is an n -dimensional simplex. It is also conjectured that equality in (1) is attained only if K is a simplex. The inequality (1) for $n = 2$ was proved by Mahler [Ma1] with the case of equality proved by Meyer [Me91]. Other cases, like e.g. bodies of revolution, were treated in [MR98]. Several special cases in the centrally symmetric setting can be found in [SR, R86, GMR, Me86, R87]. Not many special cases in which (1) is true seem to be known, one such is proved in [MR06]: all n dimensional polytopes with at most $n + 3$ vertices (or facets). For more information on Mahler's conjecture, see an expository article [Tao] by Tao.

The (non-exact) reverse Santaló inequality of Bourgain and Milman [BM] is

$$\mathcal{P}(K) \geq c^n \mathcal{P}(B_2^n)$$

where c is a positive constant and B_2^n is the Euclidean ball (or any ellipsoid) in \mathbb{R}^n . Kuperberg [Ku] reproved this result with an improved constant. This should be compared with the Blaschke-Santaló inequality

$$\mathcal{P}(K) \leq \mathcal{P}(B_2^n)$$

with equality only for ellipsoids ([S], [P], see [MP] or also [MR06] for a simple proof of both the inequality and the case of equality)

The volume product is affinely invariant, that is, $\mathcal{P}(A(K)) = \mathcal{P}(K)$ for every affine isomorphism $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$. Thus, in order to deal with local behavior of the volume product we need the following affine-invariant (the *Banach-Mazur*) distance between convex bodies:

$$d_{BM}(K, L) = \inf \left\{ c : A(L) \subset B(K) \subset cA(L), \text{for affine isomorphisms } A, B \text{ on } \mathbb{R}^n \right\}$$

If both K and L are symmetric convex bodies, this is just the classical Banach-Mazur distance.

In a recent paper [NPRZ], the following result, connected to the symmetric form of Mahler's conjecture, is proved:

Theorem [NPRZ] *Let K be an origin-symmetric convex body in \mathbb{R}^n . Then*

$$\mathcal{P}(K) \geq \mathcal{P}(B_\infty^n),$$

provided that $d_{BM}(K, B_\infty^n) \leq 1 + \delta$, and $\delta = \delta(n) > 0$ is small enough (where B_∞^n is the ℓ_∞^n unit ball). Moreover, the equality holds only if $d_{BM}(K, B_\infty^n) = 1$, i.e., if K is a parallelopiped.

In this paper we prove the analogous result for the n -dimensional simplex.

Theorem 1. *There exists $\delta(n) > 0$ such that the following holds: Let S be a simplex in \mathbb{R}^n and K a convex body in \mathbb{R}^n with $d_{BM}(K, S) = 1 + \delta$ for $0 < \delta < \delta(n)$. Then*

$$\mathcal{P}(K) \geq \mathcal{P}(S) + C\delta,$$

where $C = C(n)$ is a positive constant.

There are some profound differences between the symmetric and the non-symmetric cases. The most important one is, perhaps, the changed location of the Santaló point when the body changes even slightly. Section 2 of this paper deals with this change (it is shown that it obeys linear stability) and its implication on the volume of the polar body (square-order stability). We believe that the results of Section 2 have importance for their own sake.

Section 3 presents the necessary changes to the methods of [NPRZ]. Among these we mention, in particular, the proof of Lemma 4 here, which is the analogue of Section 5 of [NPRZ]. This Lemma required a new proof that will work also in the non-symmetric setting. We provide here a coordinate-free proof that has a potential of being useful in other settings as well.

From now on we fix a regular simplex Δ_n to be the convex hull of $n + 1$ vertices v_0, \dots, v_n where v_0, \dots, v_n are points on \mathbb{S}^{n-1} satisfying

$$\langle v_i, v_j \rangle = \begin{cases} 1, & \text{if } i = j \\ -\frac{1}{n}, & \text{otherwise.} \end{cases}$$

Note that $\Delta_n^\circ = -n\Delta_n$. Most of the constants throughout the proofs depend on the dimension n . They do not depend on the body K . We use the same letter (usually C , c etc.) to denote different constants in different paragraphs or even in different lines.

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2 Continuity of the Santaló map

The following volume formula is known (using polar coordinates). For every interior point z of K ,

$$|K^z| = \frac{1}{n} \int_{\mathbb{S}^{n-1}} (h_K(\theta) - \langle z, \theta \rangle)^{-n} d\sigma(\theta).$$

where σ is the spherical Lebesgue measure and h_K is the support function of K . By the minimum property of $|K^z|$ at the Santaló point $s(K)$, it turns out that $z = s(K)$ is a *unique* point satisfying the condition (see [Sc])

$$\int_{\mathbb{S}^{n-1}} (h_K(\theta) - \langle z, \theta \rangle)^{-n-1} \theta d\sigma(\theta) = 0. \quad (2)$$

This is equivalent to the fact that the centroid of K^z is the origin.

Denote by \mathcal{K}^n the space of convex bodies in \mathbb{R}^n endowed with the Hausdorff metric d_H . The space (\mathcal{K}^n, d_H) is isometrically embedded in the space $C(\mathbb{S}^{n-1})$ of continuous functions on the sphere \mathbb{S}^{n-1} by the isometry $K \mapsto h_K$, that is, $d_H(K, L) = \|h_K - h_L\|_\infty$ for every K, L in \mathcal{K}^n .

Proposition 1. *The Santaló map $s : (\mathcal{K}^n, d_H) \rightarrow \mathbb{R}^n$ is continuous. Furthermore, for every convex body K_0 , there exist positive constants $C = C(K_0)$ and $\delta = \delta(K_0)$ such that*

$$d_H(K, K_0) \leq \delta \Rightarrow |s(K) - s(K_0)| \leq C d_H(K, K_0).$$

Proof. The continuity of $s(K)$ is proved using a standard argument (the dominated convergence theorem and the uniqueness of $s(K)$ in (2)).

For the second part, fix a convex body K_0 and let K be any convex body which is close to K_0 in the Hausdorff metric. Since $s(K_0)$ is in the interior of K_0 , there is a $r_0 > 0$ such that the ball $B(s(K_0), r_0)$ with center $s(K_0)$ and radius r_0 is contained in K_0 . Then, since

$$K_0^{s(K_0)} = (K_0 - s(K_0))^\circ \subset B(0, r_0)^\circ = B(0, 1/r_0),$$

we have, for every $\theta \in \mathbb{S}^{n-1}$,

$$h_{K_0}(\theta) - \langle s(K_0), \theta \rangle = \|\theta\|_{K_0^{s(K_0)}} \geq r_0.$$

Define three functions a, x, y on \mathbb{S}^{n-1} by

$$\begin{aligned} a(\theta) &= h_{K_0}(\theta) - \langle s(K_0), \theta \rangle, \\ x(\theta) &= \langle s(K), \theta \rangle - \langle s(K_0), \theta \rangle \quad \text{and} \\ y(\theta) &= h_K(\theta) - h_{K_0}(\theta). \end{aligned}$$

Note that, for every $\theta \in \mathbb{S}^{n-1}$,

$$|a(\theta)| \geq r_0, \quad |x(\theta)| \leq |s(K) - s(K_0)|, \quad |y(\theta)| \leq d_H(K, K_0)$$

and $h_K(\theta) - \langle s(K), \theta \rangle = a(\theta) - x(\theta) + y(\theta)$. By the Talyor formula, we can write as

$$\begin{aligned} (a - x + y)^{-n-1} &= a^{-n-1} \left[1 - \frac{x - y}{a} \right]^{-n-1} \\ &= a^{-n-1} \left[1 + (n+1) \frac{x - y}{a} + f\left(\frac{x - y}{a}\right) \right] \end{aligned}$$

where $f(t) := (1-t)^{-n-1} - 1 - (n+1)t$ is $O(t^2)$ for small t . Then (2) implies

$$\begin{aligned} 0 &= \int_{\mathbb{S}^{n-1}} (h_K(\theta) - \langle s(K), \theta \rangle)^{-n-1} \theta d\sigma(\theta) = \int_{\mathbb{S}^{n-1}} (a - x + y)^{-n-1} \theta d\sigma(\theta) \\ &= \int_{\mathbb{S}^{n-1}} \left(a^{-n-1} + \frac{(n+1)(x-y)}{a^{n+2}} + \frac{f\left(\frac{x-y}{a}\right)}{a^{n+1}} \right) \theta d\sigma(\theta) \\ &=: (1) + (n+1) \left[(2) - (3) \right] + (4) \end{aligned}$$

where (1), (2), (3) and (4) are:

(1)

$$\int_{\mathbb{S}^{n-1}} a(\theta)^{-n-1} \theta d\sigma(\theta) = \int_{\mathbb{S}^{n-1}} (h_{K_0}(\theta) - \langle s(K_0), \theta \rangle)^{-n-1} \theta d\sigma(\theta) = 0.$$

(2)

$$\begin{aligned} \left| \int_{\mathbb{S}^{n-1}} \frac{x(\theta)}{a(\theta)^{n+2}} \theta d\sigma(\theta) \right| &= \left| \int_{\mathbb{S}^{n-1}} \frac{\langle s(K) - s(K_0), \theta \rangle}{[h_{K_0}(\theta) - \langle s(K_0), \theta \rangle]^{n+2}} \theta d\sigma(\theta) \right| \\ &\geq \left\langle \zeta, \int_{\mathbb{S}^{n-1}} \frac{\langle s(K) - s(K_0), \theta \rangle}{[h_{K_0}(\theta) - \langle s(K_0), \theta \rangle]^{n+2}} \theta d\sigma(\theta) \right\rangle \\ &= |s(K) - s(K_0)| \int_{\mathbb{S}^{n-1}} \frac{|\langle \zeta, \theta \rangle|^2 d\sigma(\theta)}{[h_{K_0}(\theta) - \langle s(K_0), \theta \rangle]^{n+2}} \\ &\geq |s(K) - s(K_0)| \int_{\mathbb{S}^{n-1}} \frac{|\langle \zeta, \theta \rangle|^2 d\sigma(\theta)}{|\text{diam}(K_0)|^{n+2}} \\ &= C_1 |s(K) - s(K_0)|, \end{aligned}$$

where $\zeta = \frac{s(K) - s(K_0)}{|s(K) - s(K_0)|} \in \mathbb{S}^{n-1}$ and $C_1 = |B_2^n| |\text{diam}(K_0)|^{-n-2}$.

(3)

$$\left| \int_{\mathbb{S}^{n-1}} \frac{y(\theta)}{a(\theta)^{n+2}} \theta d\sigma(\theta) \right| \leq \int_{\mathbb{S}^{n-1}} \frac{|y(\theta)|}{|a(\theta)|^{n+2}} |\theta| d\sigma(\theta) \leq \frac{1}{r_0^{n+2}} d_H(K, K_0).$$

(4)

$$\begin{aligned}
\left| \int_{\mathbb{S}^{n-1}} \frac{f\left(\frac{x-y}{a}\right)}{a^{n+1}} \theta d\sigma(\theta) \right| &\leq \int_{\mathbb{S}^{n-1}} \frac{c \left| \frac{x(\theta)-y(\theta)}{a(\theta)} \right|^2}{|a(\theta)|^{n+1}} |\theta| d\sigma(\theta) \\
&\leq 2c \int_{\mathbb{S}^{n-1}} \frac{|x(\theta)|^2 + |y(\theta)|^2}{|a(\theta)|^{n+3}} d\sigma(\theta) \\
&\leq 2c \int_{\mathbb{S}^{n-1}} \frac{|s(K) - s(K_0)|^2 + d_H(K, K_0)^2}{r_0^{n+3}} d\sigma(\theta) \\
&= C_2 \left(|s(K) - s(K_0)|^2 + d_H(K, K_0)^2 \right),
\end{aligned}$$

where C is an absolute constant such that $|f(t)| \leq C|t|^2$ for t near 0 and $C_2 = 2cn |B_2^n| r_0^{-n-3}$.

Finally we have

$$\begin{aligned}
C_1 |s(K) - s(K_0)| &\leq |(2)| = \left| (3) - \frac{1}{n+1} ((1) + (4)) \right| \\
&\leq \frac{1}{r_0^{n+2}} d_H(K, K_0) + \frac{C_2}{n+1} \left(|s(K) - s(K_0)|^2 + d_H(K, K_0)^2 \right).
\end{aligned}$$

By continuity of $s(K)$ (and, in fact, local uniform continuity at K_0), $|s(K) - s(K_0)| \rightarrow 0$ whenever $d_H(K, K_0) \rightarrow 0$. Thus the two quadratic terms in the inequality above can be ignored whenever δ is small enough. Therefore

$$|s(K) - s(K_0)| \leq C d_H(K, K_0)$$

where C is a constant greater than $|\text{diam}(K_0)|^{n+2} |B_2^n|^{-1} r_0^{-n-2}$.

□

Proposition 2. *Let K be a convex body in \mathbb{R}^n . If $z \in \text{int}(K)$ is close enough to $s(K)$ then*

$$|K^z| \leq |K^{s(K)}| \left(1 + \frac{c}{r_0^2} |z - s(K)|^2 \right),$$

where $r_0 > 0$ is such that $B(s(K), r_0) \subset K$ and c is a constant independent of K .

Proof. Assume that 0 is the Santaló point of K . Then $K^{s(K)} = K^\circ$,

$$|K^\circ| = \int_{K^\circ} dy \quad \text{and} \quad |K^z| = \int_{K^\circ} \frac{dy}{(1 - \langle z, y \rangle)^{n+1}}$$

(cf. e.g. Lemma 3 of [MW]). Note that $K^\circ \subset B(0, r_0^{-1})$, Hence $|\langle z, y \rangle| \leq \frac{|z|}{r_0}$ for $y \in K^\circ$. We represent $(1 - t)^{-(n+1)}$ as $1 + (n+1)t + g(t)$. We have

$$\int_{K^\circ} \langle z, y \rangle dy = 0$$

because 0 is the Santaló point of K . Thus

$$|K^z| = |K^\circ| + \int_{K^\circ} g(\langle z, y \rangle) dy$$

and we get

$$|K^z| \leq |K^\circ| \left(1 + \sum_{j=2}^{\infty} \frac{(n+1)(n+2)\dots(n+j)}{j!} \left(\frac{|z|}{r_0} \right)^j \right).$$

We finally have

$$|K^z| \leq |K^\circ| \left(1 + c \left(\frac{|z|}{r_0} \right)^2 \right)$$

if $|z| \leq \frac{r_0}{2}$ (say). □

Remark 1. Under the assumptions of Proposition 2, if $|z - s(K)| < r_0$ and K contains a ball $B(z, 2r_0)$ then it contains $B(s(K), r_0)$ this will be used later in the application of Proposition 2.

3 Construction of auxiliary Polytopes

In this section we prove an analogue of [NPRZ] for the n -dimensional simplex. Thus most of the ideas and tools that are used in the proofs in this section are basically adaptations of those from [NPRZ]. Lemma 4, which replaces Section 4 of [NPRZ], had to be worked out anew and to be put on a less coordinate dependent basis.

Let F be a k -dimensional face of Δ_n for $0 \leq k < n$ and denote by c_F its centroid. Consider the affine subspace $H_F = c_F + F^\perp$ where $F^\perp = \{y \in \mathbb{R}^n : \langle x, y \rangle = 0 \forall x \in F\}$. Take a $t > 0$ such that tH_F is tangent to K . In case that $(1-\delta)\Delta_n \subset K \subset \Delta_n$, it should be $1-\delta \leq t \leq 1$. Let x_F be such a tangent point, that is, $x_F \in tH_F \cap \partial K$ and put $y_F = tc_F$. Denote the dual face of F by $F^* = \{y \in \Delta_n^\circ : \langle x, y \rangle = 1 \forall x \in F\}$. By the same way as above, we have points x_F^* and y_F^* by replacing F , K and Δ_n by F^* , K° and Δ_n° , respectively. These four points x_F , y_F , x_F^* and y_F^* have the following properties.

Lemma 1. Let F be a face of Δ_n . Suppose that $(1-\delta)\Delta_n \subset K \subset \Delta_n$. Then

1. $\langle x_F, x_F^* \rangle = 1 = \langle y_F, y_F^* \rangle$
2. $\langle x_F - y_F, c_F \rangle = 0 = \langle x_F^* - y_F^*, c_F \rangle$
3. $|x_F - y_F| < 2\delta$ and $|x_F^* - y_F^*| < 2n\delta$.

Proof. 1. Let $t, s > 0$ be such that $x_F \in tH_F \cap \partial K$ and $x_F^* \in sH_{F^*} \cap \partial K^\circ$. Then we can check easily that

$$\begin{aligned} H_{F^*} &:= c_{F^*} + (F^*)^\perp \\ &= \{z \in \mathbb{R}^n : \langle z, h \rangle = 1 \forall h \in H_F\}. \end{aligned}$$

Consider a hyperplane G containing tH_F which is tangent to K at x_F and let α be the dual point of G , that is, $\langle \alpha, z \rangle = 1$ for every $z \in G$. So $\langle \alpha, th \rangle = 1$ for all $h \in H_F$ which implies $t\alpha \in H_{F^*}$. Since $\alpha \in \partial K^\circ$ by construction of G , we get $\alpha \in \frac{1}{t}H_{F^*} \cap \partial K^\circ$ which implies $s = 1/t$. After all, $\langle x_F, x_F^* \rangle = 1$ since $x_F \in tH_F$ and $x_F^* \in \frac{1}{t}H_{F^*}$, and $\langle y_F, y_F^* \rangle = \langle tc_F, \frac{1}{t}c_{F^*} \rangle = 1$.

2. Note that $\langle x_F - y_F, c_F \rangle = \langle x_F, c_F \rangle - \langle y_F, c_F \rangle = t - t = 0$. Similarly, we have $\langle x_F^* - y_F^*, c_{F^*} \rangle = 0$. Thus $\langle x_F^* - y_F^*, c_F \rangle = 0$ since $c_{F^*} = \frac{1}{|c_F|^2}c_F$.

3. Write $F = \text{conv}(v_0, \dots, v_k)$. Then F^\perp is in the linear span of v_{k+1}, \dots, v_n and hence $tH_F = tc_F + F^\perp$ should be in the linear span of v_{k+1}, \dots, v_n and c_F . Thus

$$\begin{aligned} tH_F \cap \Delta_n &= (tc_F + F^\perp) \cap \text{conv}(v_0, v_1, \dots, v_n) \\ &= (tc_F + F^\perp) \cap \text{conv}(c_F, v_{k+1}, \dots, v_n) \\ &\subset tc_F + (1-t)\text{conv}(v_{k+1}, \dots, v_n). \end{aligned}$$

Therefore

$$\begin{aligned} |x_F - y_F| &\leq \text{diam}(tH_F \cap \Delta_n) \leq (1-t)\text{diam}(\text{conv}(v_{k+1}, \dots, v_n)) \\ &\leq \text{diam}(\Delta_n) \delta. \end{aligned}$$

Similarly, we get $|x_F^* - y_F^*| \leq \text{diam}(\Delta_n^\circ) \delta$. □

Let \mathcal{F} be the set of all faces of Δ_n . A family \mathbb{F} of n faces F_0, \dots, F_{n-1} in \mathcal{F} is called a *flag* over \mathcal{F} if each F_k is a k -dimensional face in \mathcal{F} and $F_0 \subset F_1 \subset \dots \subset F_{n-1}$.

For each face $F \in \mathcal{F}$, we constructed four points x_F, x_F^*, y_F and y_F^* in the previous paragraph. These points induce the following four polytopes (in general, not convex):

$$\begin{aligned} P &= \bigcup_{\mathbb{F}} \text{conv}(0, x_{F_0}, \dots, x_{F_{n-1}}), \quad P' = \bigcup_{\mathbb{F}} \text{conv}(0, x_{F_0}^*, \dots, x_{F_{n-1}}^*), \\ Q &= \bigcup_{\mathbb{F}} \text{conv}(0, y_{F_0}, \dots, y_{F_{n-1}}), \quad Q' = \bigcup_{\mathbb{F}} \text{conv}(0, y_{F_0}^*, \dots, y_{F_{n-1}}^*) \end{aligned}$$

where $\mathbb{F} := \{F_0, \dots, F_{n-1}\}$ runs over all flags of \mathcal{F} . Under the assumption $(1-\delta)\Delta_n \subset K \subset \Delta_n$, they clearly satisfy $P \subset K$, $P' \subset K^\circ$, $(1-\delta)\Delta_n \subset Q \subset \Delta_n$ and $\Delta_n^\circ \subset Q' \subset \frac{1}{1-\delta}\Delta_n^\circ$.

Lemma 2. $|Q| \cdot |Q'| \geq |\Delta_n| \cdot |\Delta_n^\circ|$

The proof is essentially the same as the proof of Lemma 7 of [NPRZ].

Lemma 3. *Suppose that $(1 - \delta)\Delta_n \subset K \subset \Delta_n$. Then there exist constants C_1 and C_2 such that $||P| - |Q|| \leq C_1\delta^2$ and $||P'| - |Q'|| \leq C_2\delta^2$*

Proof. We can check that Lemma 4 of [NPRZ] is also true for the simplex Δ_n . This fact, together with Lemma 1 here and Lemma 5 of [NPRZ] (taking $X_0 = \{c_F\}$, $X_1 = \{x_F\}$, $X_2 = \{y_F\}$ and similarly for the starred points), completes the proof of the lemma. \square

Suppose that all the centroids of facets of Δ_n belong to K . Then, for every facet F of Δ_n ,

$$x_F = y_F = c_F \quad \text{and} \quad x_F^* = y_F^* = c_{F^*}.$$

This is helpful in the proof of the following lemma.

Lemma 4. *There exists $c' > 0$ such that if $\delta = \min\{d > 0 : (1 - d)\Delta_n \subset K \subset \Delta_n\}$ is small enough and if all the centroids of facets of Δ_n belong to K , then $|K| \geq |P| + c'\delta$ or $|K^\circ| \geq |P'| + c'\delta$.*

Proof. We begin by proving that there exists a constant $c_1 > 0$ such that $(1 + c_1\delta)P' \not\subset P^\circ$. Since δ is the minimal number that satisfies $(1 - \delta)\Delta_n \subset K$, we can find a vertex v_j of Δ_n , say v_0 , such that $(1 - \delta)v_0 \in \partial K$. Taking $F_0 = \{v_0\}$ in Lemma 1, we conclude the existence of

$$x_0 = tv_0 + h \in \partial K, \quad \text{with } h \in v_0^\perp, \quad |h| < C\delta, \quad 1 - \delta \leq t \leq 1,$$

and

$$x_0^* = sv_0 \in \partial K^\circ, \quad \text{with } 1 \leq s \leq \frac{1}{1 - \delta},$$

such that $\langle x_0, x_0^* \rangle = ts = 1$.

Let $z^* \in \partial K^\circ$ be such that $\langle z^*, (1 - \delta)v_0 \rangle = 1$. Then $H = \{x ; \langle z^*, x \rangle = 1\}$ is a support hyperplane of K at $(1 - \delta)v_0$. Thus $\langle x_0, z^* \rangle \leq 1$. Since H is also a support hyperplane of $(1 - \delta)\Delta_n$ at $(1 - \delta)v_0$, it follows that x_0 lies below the one-sided cone \mathcal{C} with vertex $(1 - \delta)v_0$, which is the complimentary half of the cone with the same vertex, spanned by $(1 - \delta)\Delta_n$. Take a typical facet G of the cone \mathcal{C} . Say $G \subset \{x ; \langle x, -nv_1 \rangle = 1 - \delta\}$. The highest point (with respect to the direction v_0) of $G \cap \Delta_n$ is the intersection of G with the line segment $[v_0, v_1]$. A simple calculation shows that this is the point $\beta v_0 + (1 - \beta)v_1$ with $\beta = 1 - \frac{\delta}{n+1}$. The height of this point is found by computing its projection on the altitude $[v_0, -\frac{v_0}{n}]$ of Δ_n . This is $\beta v_0 + (1 - \beta)(-\frac{v_0}{n}) = (1 - \frac{\delta}{n})v_0$. we conclude that

$$t \leq 1 - \frac{\delta}{n} \quad \text{and} \quad s \geq \frac{1}{1 - \frac{\delta}{n}}.$$

Thus

$$x_0^* = sv_0 \text{ with } \frac{1}{1 - \frac{\delta}{n}} \leq s \leq \frac{1}{1 - \delta}.$$

We look now at the vector $h \in v_0^\perp$ that was found above (with $x_0 = tv_0 + h$). There exists one of the vectors $-nv_j - v_0$, $j = 1, \dots, n$, which are vertices in v_0^\perp of a regular simplex with center 0, such that $\langle -nv_j - v_0, h \rangle \leq 0$. We may assume that this j is 1 and denote $v = \gamma(-nv_1 - v_0) \in v_0^\perp$ for some $\gamma > 0$ whose size will be determined later. Note that $|v| = \gamma\sqrt{n^2 - 1}$ and that $\langle v, h \rangle \leq 0$.

Define $\tilde{x} = x_0^* + v$. We claim that if γ is chosen correctly then $\tilde{x} \in P^\circ$ and, for some $c_1 > 0$, $\tilde{x} \notin (1 + c_1)P'$. To verify that $\tilde{x} \in P^\circ$ we have to check that $\langle \tilde{x}, x_F \rangle \leq 1$ for all the vertices x_F of P . These vertices are of the form $x_F = \frac{1}{k} \sum_{i=1}^k v_{j_i} + g$, $1 \leq k \leq n$, $|g| < C\delta$ and $g = 0$ if $k = n$.

- 1) Let $x_{F_0} = x_0 = tv_0 + h$. Then $\langle \tilde{x}, x_{F_0} \rangle = \langle x_0^*, x_0 \rangle + \langle v, x_0 \rangle = 1 + \langle v, h \rangle \leq 1$.
- 2) Let $x_F = \frac{1}{k} \sum_{i=1}^k v_{j_i} + g$, $1 \leq k \leq n$, $|g| < C\delta$. Assume first that the index 0 is not among the j_i -s. Say $x_F = \frac{1}{k} \sum_{j=1}^k v_j + g$ (it is true that v_1 plays a somewhat different role than the other indices $j \geq 1$, but the result of the coming evaluation comes up to be the same). Then

$$\langle \tilde{x}, x_F \rangle = \frac{1}{k} \sum_{j=1}^k s \langle v_0, v_j \rangle + \langle v, \frac{1}{k} \sum_{j=1}^k v_j \rangle + \langle sv_0, g \rangle + \langle v, g \rangle.$$

We have $|\frac{1}{k} \sum_{j=1}^k v_j| = \sqrt{\frac{n-k+1}{nk}}$ thus

$$\langle \tilde{x}, x_F \rangle \leq s \left(-\frac{1}{n} + C\delta \right) + |v| \left(\sqrt{\frac{n-k+1}{nk}} + C\delta \right) < 1$$

for small δ , if $\gamma < \frac{c_2}{n}$ for an appropriate constant c_2 .

- 3) Let x_F be as in 2) above, now with 0 among the j_i -s, say $x_F = \frac{1}{k} \sum_{j=0}^{k-1} v_j + g$ (same remark about v_1). The calculation now gives

$$\begin{aligned} \langle \tilde{x}, x_F \rangle &\leq \frac{s}{k} - \frac{s(k-1)}{nk} + |v| \left| \frac{1}{k} \sum_{j=0}^{k-1} v_j \right| + (s + |v|)|g| \leq \\ &s \left(\frac{1}{k} - \frac{k-1}{kn} + C\delta \right) + |v| \left(\sqrt{\frac{n-k+1}{nk}} + C\delta \right) < 1 \end{aligned}$$

if δ is small and $\gamma \leq \frac{c_2}{n}$ (note that in this case $k \geq 2$).

We fix now the constant γ that was introduced above to be precisely $\frac{c_2}{n}$ with the constant c_2 obtained above. Then $\tilde{x} \in P^\circ$ (provided that δ is small enough). As \tilde{x} is a positive linear combination of $x_0^* = sv_0$ and $-nv_1$, the line segment connecting the origin to \tilde{x} must cross the edge $[sv_0, -nv_1]$ of P' . Thus we look for $M > 0$ and $0 < \theta < 1$ such that the equality

$$sv_0 + v = M(\theta sv_0 + (1 - \theta)(-nv_1))$$

will hold. Substituting $v = \gamma(-nv_1 - v_0)$ we get $M = 1 + \gamma(1 - \frac{1}{s})$. As we had the evaluation $\frac{1}{s} = t \leq 1 - \frac{\delta}{n}$, we get

$$M \geq 1 + \frac{\gamma}{n}\delta = 1 + \frac{c_2}{n^2}\delta.$$

That is, if $c_1 < \frac{c_2}{n^2}$ then $\tilde{x} \notin (1 + c_1\delta)P'$ and we get $(1 + c_1\delta)P' \not\supseteq P^\circ$.

Assume that $K \subset (1 + \frac{c_1}{2}\delta)\text{conv}(P)$. Then $(1 - \frac{c_1}{2}\delta)P^\circ \subset \frac{1}{1+\frac{c_1}{2}\delta}P^\circ \subset K^\circ$. Let

$$\tilde{x} = (1 - \frac{c_1}{2}\delta)\tilde{x} \in (1 - \frac{c_1}{2}\delta)P^\circ \subset K^\circ.$$

By the preceding paragraph we have $\tilde{x} \notin (1 + c_1\delta)(1 - \frac{c_1}{2}\delta)P'$. As $(1 + c_1\delta)(1 - \frac{c_1}{2}\delta) > 1 + \frac{c_1}{4}\delta$ if $\delta < \frac{1}{2c_1}$, we conclude that for δ small enough, either $K \not\subset (1 + \frac{c_1}{2}\delta)\text{conv}(P)$; in which case, by Lemma 2 of [NPRZ], $|K| \geq |P| + c_3\delta$; or there exists $\tilde{x} \in K^\circ$, such that the line segment $[0, \tilde{x}]$ intersects the edge $[x_0^*, -nv_1]$ of P' , but $\tilde{x} \notin (1 + \frac{c_1}{4}\delta)P'$. That is, $K^\circ \not\subset (1 + c_4\delta)P'$. Lemma 2 of [NPRZ] completes now the proof. We remark, that, since P' is, in general, not convex the assumption of the uniform lower bound on the $(n - 1)$ -dimensional volume of its facets should be verified using the δ -approximation. \square

Proposition 3. *Let K be a convex body in \mathbb{R}^n which is close to Δ_n in the sense that $\delta = \min\{d > 0 : (1 - d)\Delta_n \subset K \subset \Delta_n\}$ is small enough. Suppose that all the centroids of facets of Δ_n belong to K . Then we have*

$$|K| |K^\circ| \geq |K| |K^\circ| \geq |\Delta_n| |\Delta_n^\circ| + C\delta.$$

Proof. We assume that $|K| \geq |P| + c\delta$ by Lemma 4. Moreover, Lemma 3 implies

$$\begin{aligned} |K| |K^\circ| &\geq (|P| + c\delta) |P'| \\ &\geq (|Q| - c_1\delta^2 + c\delta)(|Q'| - c_2\delta^2) \\ &= |Q| |Q'| + |Q'| (c\delta - c_1\delta^2) - c_2 |Q| \delta^2 - c_2\delta^2(c\delta - c_1\delta^2) \\ &\geq |Q| |Q'| + |\Delta_n^\circ| (c\delta - c_1\delta^2) - c_2 |\Delta_n| \delta^2 - c_2\delta^2(c\delta - c_1\delta^2) \end{aligned}$$

Since δ is small enough, the above inequality implies that $|K| |K^\circ| \geq |Q| |Q'| + C\delta$ for a constant $0 < C < |\Delta_n^\circ| c$. Finally, Lemma 2 completes the proof. \square

4 Proof of Theorem 1

For the proof of main theorem, let us start with the following lemma.

Lemma 5. *Let L be a convex body in \mathbb{R}^n containing the origin. Then, for every convex body K with $d_{BM}(K, L) < 1 + \delta$, there are a constant $C = C(L)$ and an affine isomorphism $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that*

$$(1 - C\delta)L \subset A(K) \subset L.$$

In particular, if $L = \Delta_n$ and $\delta > 0$ is small enough, then such C and A can be chosen to satisfy that every centroid of facets of L belongs to $A(K)$.

Proof. By definition, there are affine isomorphisms $A, B : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

$$(1 - \delta)A(L) \subset B(K) \subset A(L).$$

Clearly $A(L)$ should contain the origin 0. Put $a = A^{-1}(0)$. Then it is in L and we can write $A(x) = T(x - a)$, ($x \in \mathbb{R}^n$) for some linear transformation T on \mathbb{R}^n . Note that, for every point x ,

$$\begin{aligned} A^{-1}((1 - \delta)A(x)) &= A^{-1}((1 - \delta)T(x - a)) = A^{-1}T((1 - \delta)x + a\delta - a) \\ &= A^{-1}A((1 - \delta)x + a\delta) = (1 - \delta)x + a\delta, \end{aligned}$$

which implies $A^{-1}((1 - \delta)A(L)) = (1 - \delta)L + a\delta$. Take a constant $c > 1$ such that $-L \subset cL$. We have

$$\begin{aligned} (1 - (1 + c)\delta)L - a\delta &\subset (1 - (1 + c)\delta)L - \delta L \\ &\subset (1 - (1 + c)\delta)L + c\delta L = (1 - \delta)L \end{aligned}$$

The above two facts imply

$$\begin{aligned} (1 - (1 + c)\delta)L &\subset A^{-1}((1 - \delta)A(L)) \\ &\subset A^{-1}B(K) \subset L. \end{aligned}$$

For the case $L = \Delta_n$, note that $-\Delta_n \subset n\Delta_n$. Thus we have

$$(1 - (n + 1)\delta)\Delta_n \subset A^{-1}B(K) \subset \Delta_n.$$

Let S be a simplex of minimal volume containing $K_1 := A^{-1}B(K)$. It is proved in [Kl] that all the centroids of facets of S belong to K_1 . On the other hand, if

$x \in S \setminus (1 + \kappa) \left((1 - (n+1)\delta) \Delta_n \right)$, then by Lemma 2 of [NPRZ],

$$\begin{aligned} |S| &\geq \left| (1 - (n+1)\delta) \Delta_n \right| + \frac{\kappa (1 - (n+1)\delta)^{n \frac{1}{n}} \left(\frac{n^2}{n+1} |\Delta_n| \right)}{n} \\ &= |\Delta_n| \left(1 - (n+1)\delta \right)^n \left(1 + \frac{\kappa}{n+1} \right). \end{aligned}$$

If $\kappa > \kappa_0 = (n+1) [(1 - (n+1)\delta)^{-n} - 1]$, the existence of such x implies $|S| > |\Delta_n|$ which is a contradiction. Hence, for $\kappa > \kappa_0$,

$$\begin{aligned} (1 - \kappa)S &\subset (1 - \kappa) \left[(1 + \kappa) (1 - (n+1)\delta) \Delta_n \right] \\ &\subset (1 - (n+1)\delta) \Delta_n \\ &\subset K_1 \subset S. \end{aligned}$$

Note also that there exists a unique affine isomorphism A_1 satisfying $S = A_1(\Delta_n)$. Applying the argument of the first part again, we have

$$(1 - (n+1)\kappa) \Delta_n \subset A_1^{-1}(K_1) \subset \Delta_n.$$

where $\kappa > (n+1) \left((1 - (n+1)\delta)^{-n} - 1 \right) \approx n(n+1)^2\delta$ if $\delta > 0$ is small enough. \square

Proof of Theorem 1. Let K be a convex body with $d_{BM}(K, S) = 1 + \delta$ for sufficiently small $\delta > 0$. By Lemma 5, (replacing S by Δ_n), there is a constant $C = C(n) > 0$ such that

$$(1 - C\delta) \Delta_n \subset A(K) \subset \Delta_n.$$

and all the centroids of facets of Δ_n belong to $A(K)$. Since the volume product is invariant under affine isomorphisms of \mathbb{R}^n , we may assume that all the centroids of facets of Δ_n belong to K . We may also “include” the constant C into δ and assume that

$$\delta := \min \{d > 0 : (1 - d) \Delta_n \subset K \subset \Delta_n\}.$$

This implies that $d_H(K, \Delta_n) \leq \delta$. From Proposition 1 we now conclude that $|s(K)| \leq c_1 \delta$ for some $c_1 > 0$. By Proposition 2, and the remark following it, we get the inequality

$$\begin{aligned} |K^\circ| &\leq |K^{s(K)}| (1 + c_2 |s(K)|^2) \\ &\leq |K^{s(K)}| (1 + c_1 c_2 \delta^2) \end{aligned}$$

for some constant $c_2 > 0$ (a-priory, c_2 would depend on the radius of a ball centered at $s(K)$ and contained in K). The remark following Proposition 2 allows us to use

instead a ball centered at 0. The relation $(1 - \delta)\Delta_n \subset K$ then allows us to use a ball contained in, say, $\frac{1}{2}\Delta_n$ instead. Thus c_2 may be considered as independent of K). Hence

$$|K^{s(K)}| \geq |K^\circ| - c_1 c_2 |K^{s(K)}| \delta^2 \geq |K^\circ| (1 - c_1 c_2 \delta^2),$$

and the volume product of K satisfies

$$\mathcal{P}(K) = |K| |K^{s(K)}| \geq |K| |K^\circ| (1 - c_1 c_2 \delta^2).$$

Proposition 3 implies that $|K| |K^\circ| \geq |\Delta_n| |\Delta_n^\circ| + c\delta$. Finally, we have

$$\begin{aligned} \mathcal{P}(K) &\geq \left(|\Delta_n| |\Delta_n^\circ| + c\delta \right) (1 - c_1 c_2 \delta^2) \\ &\geq \mathcal{P}(S) + C\delta. \end{aligned}$$

for sufficiently small $\delta > 0$ and a constant $C > 0$. \square

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